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Constitutive Equations of Composite Laminated One-Way Panels

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Introduction

A TWO-DIMENSIONAL panel with two opposite free edges in the longitudinal direction and support along the other pair of edges can be replaced by an equivalent one-dimensional model, namely, a one-way panel element. Obviously, the advantage of this equivalent element is that it involves only one independent variable (the longitudinal coordinate). However, for a composite laminate, compliance with the boundary conditions in the transverse direction is questionable.

For composite laminated one-way panels, the two well-known versions of the constitutive model, plane-strain and plane-stress (for large and small width-to-length ratios, respectively), which are almost the same for the isotropic case, are used only as upper and lower bounds. Because of the significant gap between them, more accurate models are called for.

Whitney's[1] cylindrical bending theory (cylW) made some improvement over the plane-strain version by narrowing the aforementioned gap. Further improvement with regard to the plane-stress version (cylN) is proposed hereafter.

Formulation

A panel element is represented by a one-dimensional model (in the x–z plane), which leaves only two dependent variables, displacement in the longitudinal u and vertical w directions, or three variables for including shear deformation effect (the additional one is the rotation ψ). The force strain relations can be written as

$$N_{xx} = a\bar{\varepsilon}_{xx} + \bar{b}\chi_{xx}, \qquad M_{xx} = \bar{b}\bar{\varepsilon}_{xx} + d\chi_{xx} \tag{1}$$

where x is the coordinate of the reference surface of the one-way panel, N_{xx} and M_{xx} are the longitudinal membrane force and bending moment, and $\bar{\varepsilon}_{xx}$ and χ_{xx} are the strain of the reference surface and the change of curvature, respectively.

For classical isotropic one-way panels $\bar{b} = 0$, and a and d are straightforward membrane and bending stiffnesses. By contrast, for a composite laminated panel element with arbitrary stacking combination and orientation a, b, and d are no longer uniquely determined, and their values depend on the applied equivalent anisotropic theory.

Different equivalent one-dimensional models [represented by a, $\bar{b}$, and d of Eq. (1)] can be derived from the classical two-dimensional laminate theory, for which the stress–strain relation for each lamina j is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}_j = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_j \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}_j \tag{2}$$

where

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}_j = \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \end{Bmatrix} + z_j \begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ 2\chi_{xy} \end{Bmatrix} \tag{3}$$

where $\{\bar{\varepsilon}\}$ and $\{\chi\}$ are the strain-of-the-reference-surface and change-of-curvature vectors, respectively, and $\bar{Q}_{ij}$ is the transformed stiffness coefficients of the laminate.

Integration over the thickness yields the following force and moment resultant of the two-dimensional laminate:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{Bmatrix} \bar{\varepsilon} \\ \chi \end{Bmatrix} \tag{4}$$

where $\{N\} = \{N_{xx}, N_{yy}, N_{xy}\}$ is the membrane force vector and $\{M\} = \{M_{xx}, M_{yy}, M_{xy}\}$ the bending moment vector. A, B, and D are matrices of order 3×3, given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_z \bar{Q}_{ij}(1, z, z^2)\,\mathrm{d}z \tag{5}$$

In the following, four equivalent one-dimensional models are considered:

1) For the plane strain model, based on the assumption of two-dimensional sides constrained, $\varepsilon_{yy_j} = \gamma_{xy_j} = 0$ for each lamina j. Plane strain is usually applicable for large width-to-length (b/l) ratios. Substitution of the constraint $\varepsilon_{yy_j} = \gamma_{xy_j} = 0$ in Eq. (2) yields

$$\sigma_{xx} = \bar{Q}_{11}\varepsilon_{xx} \tag{6}$$

where

$$(a, \bar{b}, d) = \int_z \bar{Q}_{11}(1, z, z^2)\,\mathrm{d}z \tag{7}$$

2) For the plane stress model, based on the assumption of two-dimensional sides free, $\sigma_{yy_j} = \tau_{xy_j} = 0$ for each lamina j. Plane stress is usually applicable for small width-to-length ratios. Substitution of the free-edge condition $\sigma_{yy_j} = \tau_{xy_j} = 0$ in Eq. (2) yields

$$\sigma_{xx} = \bar{Q}\varepsilon_{xx} \tag{8}$$

where

$$\bar{Q} = \bar{Q}_{11} + c_1\bar{Q}_{12} + c_2\bar{Q}_{13}, \qquad c_1 = \frac{\bar{Q}_{13}\bar{Q}_{23} - \bar{Q}_{33}\bar{Q}_{12}}{\bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2}$$

$$c_2 \frac{\bar{Q}_{12}\bar{Q}_{23} - \bar{Q}_{22}\bar{Q}_{13}}{\bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2} \tag{9}$$

and a, $\bar{b}$, and d are obtained by replacing $\bar{Q}_{11}$ in Eq. (7) by $\bar{Q}$.

3) The cylindrical bending, wide beam (cylW) model is a more accurate model than the plane-strain version and, with the fiber orientation taken into account, is derived on the assumption that the displacement vector of the two-dimensional model is a function of the longitudinal coordinate alone (see Refs. 1 and 2 for panels without shear effect):

$$u(x, y, z) = \bar{u}(x) + z\psi(x), \qquad v(x, y, z) = \bar{v}(x)$$

$$w(x, y, z) = \bar{w}(x) \tag{10}$$

where u, v, and w are the displacement functions in the x, y, and z directions of the two-dimensional model.

The appropriate kinematic relations based on the preceding assumptions (which include the shear deformation effect) are

$$\bar{\varepsilon}_{xx} = \bar{u}_{,x} + \tfrac{1}{2}\bar{w}_{,x}^2, \qquad \bar{\varepsilon}_{yy} = \bar{\gamma}_{yz} = \chi_{yy} = \chi_{xy} = 0$$

$$\bar{\gamma}_{xy} = \bar{v}_{,x}, \qquad \bar{\gamma}_{xz} = \bar{w}_{,x} + \psi, \qquad \chi_{xx} = \psi_{,x} \tag{11}$$

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The variational principle [based on Eqs. (11)] yields the following equilibrium equations:

$$N_{xx,x} + q_{xx} = 0, \qquad N_{xy,x} = 0$$

$$Q_{xz,x} + (N_{xx}w_{,x})_{,x} + q_{zz} = 0, \qquad M_{xx,x} - Q_{xz} + m = 0 \quad (12)$$

with q_{xx}, q_{zz}, and m being the external applied longitudinal load, transverse load, and moment, respectively. Note that the equilibrium conditions of the equivalent one-dimensional model [Eqs. (12)] contain the additional equation

$$N_{xy,x} = 0 \quad (13)$$

For the free edge it leads to $N_{xy} = 0$, read in terms of $\{\bar{\varepsilon}\}$ and $\{\chi\}$ [see Eq. (4)]:

$$A_{13}\bar{\varepsilon}_{xx} + A_{33}\bar{v}_{,x} + B_{13}\psi_{,x} = 0 \quad (14)$$

Eliminating $\bar{v}_{,x}$ from Eq. (14) and using Eqs. (4) and (12), the following relevant stiffness coefficients are obtained:

$$a = \left(A_{11}A_{33} - A_{13}^2\right)\big/ A_{33}, \qquad \bar{b} = (B_{11}A_{33} - A_{13}B_{13})/A_{33}$$

$$d = \left(D_{11}A_{33} - B_{13}^2\right)\big/ A_{33} \quad (15)$$

Note that the stiffness coefficients involve the terms 11, 33, and 13.

4) The cylindrical bending, narrow panel (beam) (cylN) model is a more accurate model than the plane-stress version and is based on inclusion of the effect of fiber orientation with the flexibility coefficients expressed, after the matrix manipulation, as derived by Sheinman[3] (see also Ref. 4).

The strain and change of curvature of the one-dimensional model can be written in terms of longitudinal force and bending moment as

$$\bar{\varepsilon}_{xx} = \alpha_1 N_{xx} + \alpha_2 M_{xx}, \qquad \chi_{xx} = \alpha_3 N_{xx} + \alpha_4 M_{xx} \quad (16)$$

Using Eqs. (4), writing the strain and change-of-curvature vectors in terms of the membrane-force and bending-moment vectors, and equating to Eq. (16), we obtain

$$\alpha_1 = \{[I + A^{-1}B(D - BA^{-1}B)^{-1}B]A^{-1}\}_{11}$$

$$\alpha_2 = \alpha_3 = \{-A^{-1}B(D - BA^{-1}B)^{-1}\}_{11}$$

$$\alpha_4 = \{(D - BA^{-1}B)^{-1}\}_{11} \quad (17)$$

The subscript 11 means the first term of the matrix. It is seen that the α are determined not only by A_{11}, B_{11}, and D_{11} but also by A_{ij}, B_{ij}, and D_{ij} $(i, j = 1, 2, 3)$. Finally, using Eqs. (17), the stiffness coefficients are determined through

$$a = \alpha_4/\eta, \qquad \bar{b} = -\alpha_2/\eta, \qquad d = \alpha_1/\eta, \qquad \eta = \alpha_1\alpha_4 - \alpha_2\alpha_3 \quad (18)$$

Results and Discussion

For isotropic one-way panels, the plane-strain and plane-stress models are close to each other [differing by $1/(1 - \nu^2)$ only], so that the differences between the four constitutive models are insignificant. By contrast, nonsymmetric composite laminated panels show marked discrepancies caused by the nonvanishing B_{ij} $(i, j = 1, 3)$ terms (the larger the terms, the wider the discrepancy). Accordingly, such a laminate with $\alpha/-\alpha$-deg layup was chosen to illustrate the proposed cylindrical bending model approach and its advantage.

The constitutive models are most effectively studied either in the frequency domain or through the buckling behavior. Here the buckling behavior of a simply supported laminated $(\alpha/-\alpha)$ panel under an external longitudinal force was chosen as the study case. The data of this case are material carbon/epoxy with moduli $E_{11} = 1.4 \times 10^{11}$ N/m^2, $E_{22} = 0.98 \times 10^{10}$ N/m^2, $G_{13} = G_{12} = 0.1 \times 10^{10}$ N/m^2, $\upsilon_{12} = 0.34$, and thickness $h = 0.00125$ m with $h_{\text{ply}} = 0.000125$ m and length-to-thickness ratio $l/h = 10$. Obviously, in such a case the effect of shear deformation ought to be

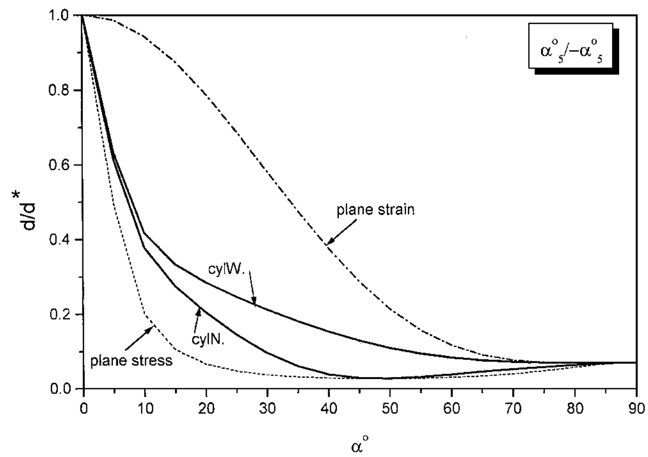

Fig. 1 Influence of fiber orientation on the bending stiffness *d* for different constitutive models in $\alpha_5/-\alpha_5$ laminate.

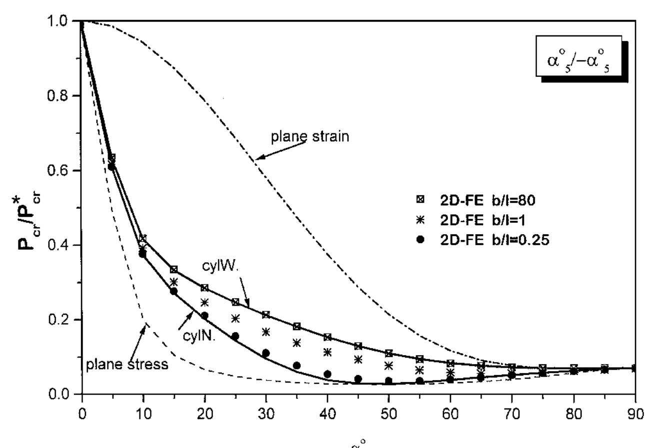

Fig. 2 Buckling load vs orientation angle of $\alpha_5/-\alpha_5$ laminate for different constitutive models compared to two-dimensional plate results.

included, but as far as the constitutive models are concerned the characteristic conclusions are the same whether it is included or not.

First, the influence of the orientation angle for stacking combination and orientation of $\alpha_5/-\alpha_5$ deg is plotted in Fig. 1 (d^* denotes the bending stiffness for $\alpha = 0$ deg). It is seen that the cylindrical bending models substantially narrow the gap between the upper (cylW) and lower (cylN) bounds. The trend to the upper or to the lower bound for a real case depends on the width-to-length b/l ratio, as illustrated in Fig. 2, in which the normalized buckling loads (P_{cr}^* denoting the buckling load for $\alpha = 0$ deg) of a sequence $\alpha_5/-\alpha_5$ deg for the different constitutive models are compared with two-dimensional results (for $b/l = 0.25$, 1.0, and 80) obtained by the plate finite element COSMOS7 code.[5] It is seen that the buckling load for the large ratio ($b/l = 80$) coincides with the cylW model and that for the small ratio with the cylN model. Note also that due to the relatively high value of B_{i3} the bounds lie far below and above the plane-strain and plane-stress curves, respectively. To bring the picture into focus, another nonsymmetric laminate, $(\alpha/-\alpha)_5$, whose B_{i3} is $\frac{1}{5}$ of that of $\alpha_5/-\alpha_5$, was considered. The results (Fig. 3) show that in this case the upper bound practically coincides with the plane-strain model (for vanishing B_{i3} they coincide completely) and that the lower bound is closer to it than in the preceding case, the gap between the bounds being wider than for $\alpha_5/-\alpha_5$. The two-dimensional results again coincide completely with cylW for the large b/l ratio and tend to cylN for a small b/l ratio. Finally, the buckling load, with shear deformation effect included and excluded, vs the width-to-length ratio, for layup of $40_5/-40_5$ deg, is given in Figs. 4 and 5 for $0 < b/l \leq 1$ and $1 \leq b/l \leq 50$, respectively. The

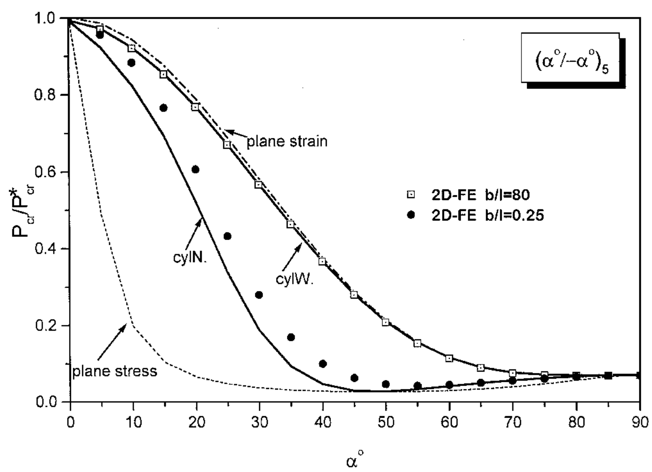

Fig. 3 Buckling load vs orientation angle of $(\alpha/-\alpha)_5$ laminate for different constitutive models compared to two-dimensional plate results.

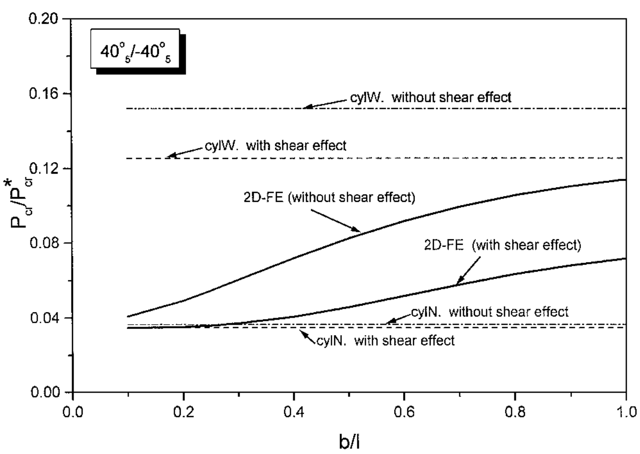

Fig. 4 Buckling load vs width-to-length ratio, $0 < b/l \leq 1$ for $40_5/-40_5$-deg laminate.

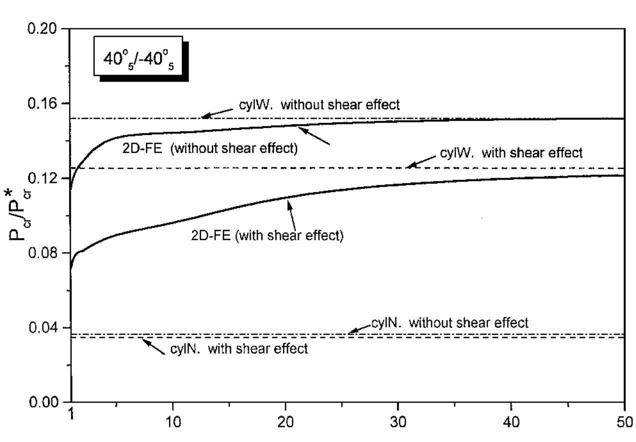

Fig. 5 Buckling load vs width-to-length ratio, $1 < b/l \leq 50$ for $40_5/-40_5$-deg laminate.

results are seen to tend to the lower bound in Fig. 4 and to the upper bound in Fig. 5.

The significant difference between the results with and without shear deformation effect is due to the low $l/h(=10)$ ratio and the high $E_{11}/G_{13}(=140)$ ratio, but the same conclusion applies to the constitutive models.

Conclusion

The well-known plane-strain and plane-stress constitutive models are unsuitable for composite laminated one-way panels. Cylindrical bending models are much more accurate and appropriate for presentation of laminated stacking combinations and orientations.

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Failure Analysis of Scarf-Patch-Repaired Carbon Fiber/Epoxy Laminates Under Compression

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Introduction

IN recent years, demands on technological development for permanent field and depot repairs of composite structures have increased considerably. As a result, repair methodologies have been developed and include a wide range of approaches, from highly refined and structurally efficient but expensive flush-patch repairs to mechanically attached metal-patch[1–3] repairs. Flush, scarf-type, bonded repairs are used on critical, highly loaded components, where load concentration and eccentricities, especially for compressive loading, must be avoided.[2]

Design methods for adhesively bonded repairs require criteria to predict both strength and durability. In this study, a three-dimensional stress analysis is performed to determine the stresses in a flush-scarf-repaired laminate under uniaxial compression (Fig. 1) so that predictions can be made of the optimum scarf angle and likely points of failure.

Scarf Joint and Optimum Scarf Angle

The joint of interest has identical adherends, uses a relatively brittle adhesive, and has small scarf angles (Fig. 2). For this simple case, the semiempirical analysis[4] predicts that the optimum scarf angle for a maximum strength joint is a function of the adhesive shear strength τ_s and laminate strength σ_{un}, given by

$$\theta_{opt} \cong \tan^{-1}(0.816\tau_s/\sigma_{un}) \tag{1}$$

For small θ, the failure stress $S_{g,f}$ of the scarf joint is determined by the maximum stress failure criterion,

$$S_{g,f} = \sigma_{un}/K_L = \tau_s/(K_A \sin\theta) \tag{2}$$

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